The uniformizability of L-topological groups

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Abstract

In this paper, we show that any L-topological group (G, τ) is uniformizable. That is, we define, using the family of prefilters which corresponds the L- neighborhood filter at the identity element of (G, τ), unique left and right invariant L- uniform structures on G compatible with the L- topology τ. On the other hand, on any group G, using a family of prefilters on G fulfills certain conditions, we construct those left and right L- uniform structures which induce a L- topology τ on G for which (G, τ) is an L-topological group and this family of prefilters coincides with the family of prefilters corresponding to the L- neighborhood filter at the identity element of (G, τ). Moreover, we show the relation between the L-topological groups and the GT_i-spaces, such as: the L- topology of an L-topological group (resp., a separated L-topological group) is completely regular (resp., $GT_{3\frac{1}{2}}$).

Keywords: Fuzzy filters; Fuzzy uniform spaces; Fuzzy topological groups; GT_i-spaces; Completely regular spaces; $GT_{3\frac{1}{2}}$-spaces; L-Tychonoff spaces.

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1. Introduction

The notion of an L-topological group (G, τ) is defined by Ahsanullah ([1]) in 1984 as an ordinary group G equipped with a L- topology τ on G such that the binary operation and the unary operation of the inverse are L- continuous with respect to τ.

In [1, 7], many results on the L-topological groups are studied. These L-topological groups are called, in [1], L- topological groups.

The L- neighborhood filter at the identity element of the L-topological group (G, τ) corresponds a family of prefilters on G ([11]). Using this family of prefilters, we construct, in this paper, a unique left invariant L- uniform structure $\mathcal{U}^l$ and a unique right invariant L- uniform structure $\mathcal{U}^r$ on G. These L- uniform structures $\mathcal{U}^l$ and $\mathcal{U}^r$ are compatible with τ, that is, $\tau_{\mathcal{U}^l} = \tau_{\mathcal{U}^r} = \tau$. This means that the L-topological group (G, τ) is uniformizable. The L- uniform structures $\mathcal{U}^l$ and $\mathcal{U}^r$ are L- uniform structures in sense of [12] which are defined as L- filters on the cartesian product $G \times G$ of G with itself.

We show also here that for any group G and any family of prefilters fulfills certain conditions , we define the left and the right L- uniform structures $\mathcal{U}^l$ and $\mathcal{U}^r$ on G such that $\tau_{\mathcal{U}^l} = \tau_{\mathcal{U}^r}$ is a L- topology τ on G for which the pair (G, τ) is an L-topological group. Moreover, this family of prefilters is exactly the family of prefilters which corresponds the L- neighborhood filter at the identity element of the L-topological group (G, τ).

Moreover, in this paper, we study some relations between the L-topological groups and the L- separation axioms GT_i which we had introduced in [2, 3, 5]. We show that the L- topology τ of an L-topological group (G, τ) is completely regular in our sense ([5]) and that the L-topological group (G, τ) is separated if and only if the L- topology τ is GT_0 (resp. GT_1, GT_2, $GT_{3\frac{1}{2}}$) if and only if the left L-uniform structure $\mathcal{U}^l$ (resp. the right L- uniform structure $\mathcal{U}^r$) is separated.

2. On L- filters

Let L be a complete chain with different least and greatest elements 0 and 1, respectively. Let $L_0 = L \setminus \{0\}$ and $L_1 = L \setminus \{1\}$. Denote by L^X the set of all L- subsets of a non-empty set X.

By a L- *filter* on X ([9, 10]) is meant a mapping $\mathcal{M} : L^X \to L$ such that $\mathcal{M}(\overline{\alpha}) \leq \alpha$ holds for all $\alpha \in L$ and $\mathcal{M}(\overline{1}) = 1$, and also $\mathcal{M}(f \wedge g) = \mathcal{M}(f) \wedge \mathcal{M}(g)$ for all $f, g \in L^X$. A L- filter $\mathcal{M}$ is called *homogeneous* if $\mathcal{M}(\overline{\alpha}) = \alpha$ for all $\alpha \in L$. If $\mathcal{M}$ and $\mathcal{N}$ are L- filters on X, $\mathcal{M}$ is said to be *finer than* $\mathcal{N}$, denoted by, $\mathcal{M} \leq \mathcal{N}$, provided $\mathcal{M}(f) \geq \mathcal{N}(f)$ holds for all $f \in L^X$. By $\mathcal{M} \not\leq \mathcal{N}$ we denote that $\mathcal{M}$ is not finer than $\mathcal{N}$.

For any set A of L- filters on X, the infimum $\bigwedge\limits_{\mathcal{M} \in A} \mathcal{M}$, with respect to the finer relation on L- filters, does not exist in general. The infimum $\bigwedge\limits_{\mathcal{M} \in A} \mathcal{M}$ of A exists *if and only if* for each non-empty finite subset $\{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$ of A we have $\mathcal{M}_1(f_1) \wedge \cdots \wedge \mathcal{M}_n(f_n) \leq \sup(f_1 \wedge \cdots \wedge f_n)$ for all $f_1, \ldots, f_n \in L^X$ ([9]). If the infimum of A exists, then for each $f \in L^X$ and n as a positive integer we have

$$(\bigwedge_{\mathcal{M} \in A} \mathcal{M})(f) = \bigvee_{\substack{f_1 \wedge \cdots \wedge f_n \leq f, \\ \mathcal{M}_1, \ldots, \mathcal{M}_n \in A}} (\mathcal{M}_1(f_1) \wedge \cdots \wedge \mathcal{M}_n(f_n)).$$

A *prefilter* on X is a non-empty subset $\mathcal{F}$ of L^X which does not contain $\overline{0}$ and closed under finite infima and super sets ([15]). For each L- filter $\mathcal{M}$ on X, the subset α-pr $\mathcal{M}$ of L^X defined by:

$$\alpha\text{-pr}\, \mathcal{M} = \{f \in L^X \mid \mathcal{M}(f) \geq \alpha\}$$

is a prefilter on X.

A *valued L- filter base* on a set X ([10]) is a family $(\mathcal{B}_\alpha)_{\alpha \in L_0}$ of non-empty subsets of L^X such that the following conditions are fulfilled:

(V1) $f \in \mathcal{B}_\alpha$ implies $\alpha \leq \sup f$.

(V2) For all $\alpha, \beta \in L_0$ and all mappings $f \in \mathcal{B}_\alpha$ and $g \in \mathcal{B}_\beta$, if even $\alpha \wedge \beta > 0$ holds, then there are a $\gamma \geq \alpha \wedge \beta$ and a L- set $h \leq f \wedge g$ such that $h \in \mathcal{B}_\gamma$.

Each valued L- filter base $(\mathcal{B}_\alpha)_{\alpha \in L_0}$ on a set X defines a L- filter $\mathcal{M}$ on X by $\mathcal{M}(f) = \bigvee\limits_{g \in \mathcal{B}_\alpha,\, g \leq f} \alpha$ for all $f \in L^X$. On the other hand, each L- filter $\mathcal{M}$ can be generated by many valued L- filter bases, and among them the greatest one $(\alpha\text{-pr}\,\mathcal{M})_{\alpha\in L_0}$.

Proposition 2.1 [10] *There is a one-to-one correspondence between the L- filters $\mathcal{M}$ on X and the families $(\mathcal{M}_\alpha)_{\alpha\in L_0}$ of prefilters on X which fulfill the following conditions:*

(1) $f \in \mathcal{M}_\alpha$ *implies* $\alpha \leq \sup f$.

(2) $0 < \alpha \leq \beta$ *implies* $\mathcal{M}_\alpha \supseteq \mathcal{M}_\beta$.

(3) *For each* $\alpha \in L_0$ *with* $\bigvee_{0<\beta<\alpha} \beta = \alpha$ *we have* $\bigcap_{0<\beta<\alpha} \mathcal{M}_\beta = \mathcal{M}_\alpha$.

This correspondence is given by $\mathcal{M}_\alpha = \alpha\text{-pr}\,\mathcal{M}$ *for all* $\alpha \in L_0$ *and* $\mathcal{M}(f) = \bigvee_{g\in\mathcal{M}_\alpha,\, g\leq f} \alpha$ *for all* $f \in L^X$.

Fuzzy neighborhood filters. In the following the L- topology τ on a set X in sense of ([8, 13]) will be used. int_τ and cl_τ denote the interior and the closure operators with respect to τ, respectively. For each L- topological space (X, τ) and each $x \in X$ the mapping $\mathcal{N}(x) : L^X \to L$ defined by

$$\mathcal{N}(x)(f) = \text{int}_\tau f(x)$$

for all $f \in L^X$ is a L- filter on X, called the L- *neighborhood filter* of the space (X, τ) at x ([11]).

$f \in L^X$ is called a τ-*neighborhood* at $x \in X$ provided $\alpha \leq \text{int}_\tau f(x)$ for some $\alpha \in L_0$. That is, f is a τ-neighborhood at x if $f \in \alpha\text{-pr}\,\mathcal{N}(x)$ for some $\alpha \in L_0$.

Let (X, τ) and (Y, σ) be two L- topological spaces. Then the mapping $f : (X, \tau) \to (Y, \sigma)$ is called L- *continuous* (or (τ, σ)-*continuous*) provided

$$\text{int}_\sigma g \circ f \leq \text{int}_\tau (g \circ f) \text{ for all } g \in L^Y.$$

3. L-topological groups

In the following we focus our study on a multiplicative group G. We denote, as usual, the identity element of G by e and the inverse of an element a of G by a^{-1}.

Let $\pi : G \times G \to G$ be a mapping defined by

$$\pi(a, b) = ab \text{ for all } a, b \in G,$$

and $i : G \to G$ a mapping defined by

$$i(a) = a^{-1} \text{ for all } a \in G,$$

that is, π and i are the binary operation and the unary operation of the inverse on G, respectively.

Here, we define the product of $f, g \in L^G$ with respect to the binary operation π on G as the L- set fg in G defined by:

$$fg = \bigwedge_{f(x)>0,\, g(y)>0} (xy)_1. \tag{3.1}$$

In particular, for all $a \in G$ and all $f \in L^G$, we have $af \in L^G$ defined by

$$af = \bigwedge_{f(x)>0} (ax)_1 \tag{3.2}$$

and $fa \in L^G$ defined by

$$fa = \bigwedge_{f(x)>0} (xa)_1 \tag{3.3}$$

Also, we can define the inverse of $f \in L^G$ with respect to the unary operation i on G as the L- set f^{-1} on G by:

$$f^{-1}(x) = f(x^{-1}) \text{ for all } x \in G. \tag{3.4}$$

The following definitions are similar to those in [14].

Definition 3.1 Let τ be a L- topology on a group G. The mapping $\pi : (G \times G, \tau \times \tau) \rightarrow (G, \tau)$ is called $(\tau \times \tau, \tau)$-*continuous in each variable separately* if for all $f \in \alpha\text{-pr}\,\mathcal{N}(ab)$, there exists $g \in \alpha\text{-pr}\,\mathcal{N}(b)$ such that $ag \leq f$ or there exists $h \in \alpha\text{-pr}\,\mathcal{N}(a)$ such that $hb \leq f$ for some $\alpha \in L_0$ and for all $a, b \in G$.

Definition 3.2 Let G be a group and τ be a L- topology on G. Then the pair (G, τ) will be called a *semi – L-topological group* if the mapping π is $(\tau \times \tau, \tau)$-continuous in each variable separately.

Definition 3.3 The mapping π is called $(\tau \times \tau, \tau)$-*continuous everywhere* if for all $f \in \alpha\text{-pr}\,\mathcal{N}(ab)$, there exist $g \in \alpha\text{-pr}\,\mathcal{N}(a)$ and $h \in \alpha\text{-pr}\,\mathcal{N}(b)$ such that $gh \leq f$ for some $\alpha \in L_0$ and for all $a, b \in G$.

Definition 3.4 The mapping i is called (τ, τ)-*continuous* if for all $f \in \alpha\text{-pr}\,\mathcal{N}(a^{-1})$, there exists an $g \in \alpha\text{-pr}\,\mathcal{N}(a)$ such that $g^{-1} \leq f$ for some $\alpha \in L_0$ and for all $a \in G$.

Definition 3.5 [1] Let G be a group and τ be a L- topology on G. Then the pair (G, τ) will be called an *L-topological group* if the mapping π is $(\tau \times \tau, \tau)$-continuous everywhere and the mapping i is (τ, τ)-continuous.

Clearly, every L-topological group is a semi – L-topological group.

Proposition 3.1 *The pair (G, τ) is an L-topological group* if and only if *for all $f \in \alpha\text{-pr}\,\mathcal{N}(a^{-1}b)$, there exist $g \in \alpha\text{-pr}\,\mathcal{N}(a)$ and $h \in \alpha\text{-pr}\,\mathcal{N}(b)$ such that $g^{-1}h \leq f$ for some $\alpha \in L_0$ and for all $a, b \in G$.*

Proof. Obvious. □

Let us call a L- set $f \in L^G$ *symmetric* if the inverse f^{-1}, defined by (3.4), fulfills that $f = f^{-1}$.

For each group G and $a \in G$, the *left* and *right translations* are the homomorphisms $l_a : G \rightarrow G$ defined by $l_a(x) = ax$ and $R_a : G \rightarrow G$ defined by $R_a(x) = xa$

for each $x \in G$, respectively. The left and right translations in L-topological groups fulfill the following result.

Proposition 3.2 [7] *Let (G, τ) be an L-topological group. Then for each $a \in G$ the left and right translations l_a and R_a are L-homeomorphisms.*

We shall use the following result.

Lemma 3.1 *Let f be an open L- set in an L-topological group (G, τ). Then for any $x_0 \in G$ the L- sets fx_0 and $x_0 f$ are also open.*

Proof. Consider the mapping

$$h : G \to G \times G, \ x \mapsto (x_0^{-1}, x)$$

and the projection mappings

$$p_1 : G \times G \to G, \ (x_1, x_2) \mapsto x_1$$

and

$$p_2 : G \times G \to G, \ (x_1, x_2) \mapsto x_2.$$

Then $(p_1 \circ h)(x) = x_0^{-1}$ and $(p_2 \circ h)(x) = x$. Since $(p_1 \circ h)$ and $(p_2 \circ h)$ are (τ, τ)-continuous, then h is also $(\tau, \tau \times \tau)$-continuous. Now, we have

$$\pi : G \times G \to G, \ (x_1, x_2) \mapsto x_1 x_2$$

is $(\tau \times \tau, \tau)$-continuous, and thus the mapping $\lambda = \pi \circ h$, for which $\lambda(x) = \pi(h(x)) = \pi(x_0^{-1}, x) = x_0^{-1} x$ for all $x \in G$, is (τ, τ)-continuous. Also, $\lambda^{-1}(x_0^{-1} x) = x$ for all $x \in G$, that is, $\lambda^{-1}(x) = x_0 x$ for all $x \in G$. In particular, $x_0 f = \lambda^{-1}(f)$ is a L- open set in (G, τ). fx_0 is also open with a similar proof. □

Recall that: If $f : X \to Y$ is a mapping between the non-empty sets X and Y and $h \in L^Y$, then the *preimage* $f^{-1}(h)$ of h with respect to f is defined by $f^{-1}(h) = h \circ f$.

Now, we prove the following result.

Lemma 3.2 *Let (G,τ) be an L-topological group and $x_0 \in G$. Then*

$$f \in \alpha\text{ -pr}\mathcal{N}(e) \ \text{ if and only if } \ x_0 f \in \alpha\text{-pr}\mathcal{N}(x_0) \ \text{ if and only if } \ f x_0 \in \alpha\text{-pr}\mathcal{N}(x_0).$$

Proof. Since the mapping $\lambda = \pi \circ h$, as in Lemma 3.1, is (τ,τ)-continuous, then $\text{int}_\tau g \circ \lambda \leq \text{int}_\tau (g \circ \lambda)$ for all $g \in L^G$. That is,

$$\text{int}_\tau f(x_0^{-1}x) = \text{int}_\tau f(\lambda(x)) \leq \text{int}_\tau (f \circ \lambda)(x) = \text{int}_\tau (\lambda^{-1}(f))(x) = \text{int}_\tau (x_0 f)(x)$$

for all $x \in G$ and all $f \in L^G$. Hence, $f \in \alpha$-pr$\mathcal{N}(e)$ if and only if $x_0 f \in \alpha$-pr$\mathcal{N}(x_0)$. The other case is similar and the proof is then complete. □

4. L-topological groups and their canonical L- uniform structures

In the sequel we show that for each L-topological group (G,τ), there are unique left and right invariant L- uniform structures on G compatible with τ.

For a family $(\mathcal{V}_\alpha)_{\alpha \in L_0}$ of subsets $\mathcal{V}_\alpha$ of L^X, consider the following conditions:

(e1) For all $\alpha \in L_0$, if $0 < \beta \leq \alpha$, then $\mathcal{V}_\alpha \subseteq \mathcal{V}_\beta$,

(e2) For all $\alpha \in L_0$ with $\bigvee\limits_{0<\beta<\alpha} \beta = \alpha$, we have $\mathcal{V}_\alpha = \bigcap\limits_{0<\beta<\alpha} \mathcal{V}_\beta$,

(e3) For all $\alpha \in L_0$ and all $f \in \mathcal{V}_\alpha$, we have $\alpha \leq \sup f$,

(e4) For all $\alpha \in L_0$ and all $f \in \mathcal{V}_\alpha$, there exists $g \in \mathcal{V}_\alpha$ such that $g^{-1} \leq f$,

(e5) For all $\alpha \in L_0$ and all $f \in \mathcal{V}_\alpha$, there exists $g \in \mathcal{V}_\alpha$ such that $gg \leq f$.

Proposition 4.1 *Let $\mathcal{N}(e)$ be the L- neighborhood filter at the identity element e of an L-topological group (G,τ). Then the family $(\alpha\text{-pr}\mathcal{N}(e))_{\alpha \in L_0}$ of prefilters α-pr$\mathcal{N}(e)$ fulfills the conditions* (e1) - (e5).

Proof. Since $0 < \beta \leq \alpha$ and $f \in \alpha\text{-pr}\,\mathcal{N}(e)$ imply that $\beta \leq \alpha \leq \text{int}_\tau f(e)$, then $f \in \beta\text{-pr}\,\mathcal{N}(e)$. Hence, $\alpha\text{-pr}\,\mathcal{N}(e) \subseteq \beta\text{-pr}\,\mathcal{N}(e)$, and (e1) is fulfilled.

From (e1), we get that $\alpha\text{-pr}\,\mathcal{N}(e) \subseteq \bigcap\limits_{0<\beta<\alpha} \beta\text{-pr}\,\mathcal{N}(e)$. Now, if $f \in \bigcap\limits_{0<\beta<\alpha} \beta\text{-pr}\,\mathcal{N}(e)$, then $f \in \beta\text{-pr}\,\mathcal{N}(e)$ for all $\beta \in L_0$ with $\alpha = \bigvee\limits_{0<\beta<\alpha} \beta$, which means that $f \in \alpha\text{-pr}\,\mathcal{N}(e)$ and hence (e2) holds.

(e3) is evident.

Since $i(e) = e^{-1} = e$ and i is (τ, τ)-continuous, then (e4) is fulfilled.

since $\pi(e, e) = ee = e$ and π is $(\tau \times \tau, \tau)$-continuous everywhere, then (e5) is fulfilled. □

Fuzzy uniform structures. Let $\mathcal{U}$ be a L- filter on $X \times X$. The *inverse* $\mathcal{U}^{-1}$ of $\mathcal{U}$ is a L- filter on $X \times X$ defined by $\mathcal{U}^{-1}(u) = \mathcal{U}(u^{-1})$ for all $u \in L^{X \times X}$, where u^{-1} is the inverse of u defined by: $u^{-1}(x, y) = u(y, x)$ for all $x, y \in X$. Let, each $\alpha \in L$, $\widetilde{\alpha}$ denote the constant mapping : $X \times X \to L$ defined by $\widetilde{\alpha}(x, y) = \alpha$ for all $x, y \in X$ ([12]).

For each pair (x, y) of elements x, y of X, the mapping $(x, y)^{\bullet} : L^{X \times X} \to L$ defined by $(x, y)^{\bullet}(u) = u(x, y)$ for all $u \in L^{X \times X}$ is a homogeneous L- filter on $X \times X$. Let $\mathcal{U}$ and $\mathcal{V}$ be L- filters on $X \times X$ such that $(x, y)^{\bullet} \leq \mathcal{U}$ and $(y, z)^{\bullet} \leq \mathcal{V}$ hold for some $x, y, z \in X$. Then the *composition* $\mathcal{V} \circ \mathcal{U}$ of $\mathcal{U}$ and $\mathcal{V}$ is ([12]) the L-filter on $X \times X$ defined by

$$(\mathcal{V} \circ \mathcal{U})(w) = \bigvee_{v \circ u \leq w} (\mathcal{U}(u) \wedge \mathcal{V}(v)) \tag{4.1}$$

for all $w \in L^{X \times X}$, where $u, v, v \circ u \in L^{X \times X}$ and

$$(v \circ u)(x, y) = \bigvee_{z \in X} (\, u(x, z) \wedge v(z, y)) \tag{4.2}$$

for all $x, y \in X$.

By a L- *uniform structure* $\mathcal{U}$ on a set X ([12]) we mean a L- filter on $X \times X$ such that:

(U1) $(x,x)^\bullet \leq \mathcal{U}$ for all $x \in X$.

(U2) $\mathcal{U} = \mathcal{U}^{-1}$.

(U3) $\mathcal{U} \circ \mathcal{U} \leq \mathcal{U}$.

A set X equipped with a L- uniform structure $\mathcal{U}$ is called a L- *uniform space*.

For any complete chain we have the following result.

Lemma 4.1 *The supremum of two L- uniform structures is a L- uniform structure.*

Proof. Clear. □

Proposition 4.2 [12] *There is a one - to - one correspondence between the L-uniform structures $\mathcal{U}$ on X and the families $(\mathcal{U}_\alpha)_{\alpha \in L_0}$ of prefilters on $X \times X$ which fulfill the following conditions:*

(u1) $0 < \beta \leq \alpha$ *implies* $\mathcal{U}_\alpha \subseteq \mathcal{U}_\beta$.

(u2) *For each* $\alpha \in L_0$ *with* $\bigvee\limits_{0<\beta<\alpha} \beta = \alpha$, *we have* $\mathcal{U}_\alpha = \bigcap\limits_{0<\beta<\alpha} \mathcal{U}_\beta$.

(u3) *For all* $\alpha \in L_0$, $u \in \mathcal{U}_\alpha$ *and* $x \in X$, *we have* $\alpha \leq u(x,x)$.

(u4) $u \in \mathcal{U}_\alpha$ *implies* $u^{-1} \in \mathcal{U}_\alpha$ *for all* $\alpha \in L_0$.

(u5) *For each* $\alpha \in L_0$ *and each* $u \in \mathcal{U}_\alpha$, *we have* $\alpha \leq \bigvee\limits_{v \in \mathcal{U}_\beta,\, v \circ v \leq u} \beta$.

This correspondence is given by $\mathcal{U}_\alpha = \alpha\text{-pr}\,\mathcal{U}$ *for all* $\alpha \in L_0$ *and* $\mathcal{U}(u) = \bigvee\limits_{v \in \mathcal{U}_\alpha,\, v \leq u} \alpha$ *for all* $u \in L^{X \times X}$.

Now we shall prove the following important results in which those conditions (e1) - (e5) for the family $(\alpha\text{-pr}\,\mathcal{N}(e))_{\alpha \in L_0}$ are necessary to construct L- uniform structures by which the L-topological group (G,τ) is uniformizable.

First, we construct these L- uniform structures and then, in another proposition, we show that (G,τ) is uniformizable.

Proposition 4.3 *Let (G,τ) be an L-topological group. Then the families $(\mathcal{U}^l_\alpha)_{\alpha\in L_0}$ and $(\mathcal{U}^r_\alpha)_{\alpha\in L_0}$ of the subsets $\mathcal{U}^l_\alpha$ and $\mathcal{U}^r_\alpha$ of $L^{G\times G}$ defined by*

$$\mathcal{U}^l_\alpha = \{u \in L^{G\times G} \mid u(x,y) = (f\wedge f^{-1})(x^{-1}y) \text{ for some } f \in \alpha\text{-pr}\,\mathcal{N}(e)\} \tag{4.3}$$

and

$$\mathcal{U}^r_\alpha = \{u \in L^{G\times G} \mid u(x,y) = (f\wedge f^{-1})(xy^{-1}) \text{ for some } f \in \alpha\text{-pr}\,\mathcal{N}(e)\} \tag{4.4}$$

correspond L- uniform structures $\mathcal{U}^l$ and $\mathcal{U}^r$ on G, respectively by the following:

$$\mathcal{U}^l_\alpha = \alpha\text{-pr}\,\mathcal{U}^l \qquad \text{and} \qquad \mathcal{U}^l(u) = \bigvee_{v\in\mathcal{U}^l_\alpha,\, v\le u} \alpha \tag{4.5}$$

and

$$\mathcal{U}^r_\alpha = \alpha\text{-pr}\,\mathcal{U}^r \qquad \text{and} \qquad \mathcal{U}^r(u) = \bigvee_{v\in\mathcal{U}^r_\alpha,\, v\le u} \alpha. \tag{4.6}$$

Proof. Since $\widetilde{0}(x,x) = 0 \neq 1 = (f\wedge f^{-1})(e) = (f\wedge f^{-1})(x^{-1}x)$ for all $f \in \alpha\text{-pr}\mathcal{N}(e)$ and all $x\in G$, then $\widetilde{0} \notin \mathcal{U}^l_\alpha$ for all $\alpha \in L_0$.

Also, $\widetilde{1} \in \mathcal{U}^l_\alpha$ for all $\alpha\in L_0$, from that there exists a symmetric L- set $f = e_1 = (x^{-1}yy^{-1}x)_1 = (x^{-1}y)_1(y^{-1}x)_1 \in \alpha\text{-pr}\mathcal{N}(e)$ such that $(f\wedge f^{-1})(x^{-1}y) = f(x^{-1}y)\wedge f(y^{-1}x) = 1$ for all $x,y\in G$.

Let $u \in \mathcal{U}^l_\alpha$ for all $\alpha \in L_0$ and $v \geq u$. Then $v(x,y) \geq (f\wedge f^{-1})(x^{-1}y)$ for some $f\in \alpha\text{-pr}\mathcal{N}(e)$ and for all $x,y\in G$. But $v \leq \widetilde{1} \in \mathcal{U}^l_\alpha$ implies that there is $g \in \alpha\text{-pr}\mathcal{N}(e)$ such that $v(x,y) \leq (g\wedge g^{-1})(x^{-1}y)$ for all $x,y\in G$. That is, there is some $h\in\alpha\text{-pr}\mathcal{N}(e)$ such that $v(x,y) = (h\wedge h^{-1})(x^{-1}y)$ for all $x,y\in G$. Hence $v\in\mathcal{U}^l_\alpha$ for all $\alpha\in L_0$.

Since $(f\wedge g)\in\alpha\text{-pr}\mathcal{N}(e)$ whenever $f\in\alpha\text{-pr}\mathcal{N}(e)$ and $g\in\alpha\text{-pr}\mathcal{N}(e)$, then for any $u,v\in\mathcal{U}^l_\alpha$, we get that

$$\begin{aligned}(u\wedge v)(x,y) &= u(x,y)\wedge v(x,y)\\ &= (f\wedge f^{-1})(x^{-1}y)\wedge(g\wedge g^{-1})(x^{-1}y) \text{ for some } f,g\in\alpha\text{-pr}\mathcal{N}(e)\\ &= ((f\wedge g)\wedge(f\wedge g)^{-1})(x^{-1}y) \text{ for some } f,g\in\alpha\text{-pr}\mathcal{N}(e).\end{aligned}$$

Hence $(u \wedge v) \in \mathcal{U}^l_\alpha$ for all $\alpha \in L_0$. Thus $\mathcal{U}^l_\alpha$ is a prefilter on $G \times G$ for all $\alpha \in L_0$.

Now, let $0 < \beta \leq \alpha$ and $u \in \mathcal{U}^l_\alpha$. Then from (e1) for the family $(\alpha\text{-pr}\,\mathcal{N}(e))_{\alpha \in L_0}$, we get that $u(x, y) = (g \wedge g^{-1})(x^{-1}y)$ for some $g \in \beta\text{-pr}\,\mathcal{N}(e)$ for all $x, y \in G$, and then $u \in \mathcal{U}^l_\beta$. Hence, the condition (u1) of Proposition 4.2 holds.

From (u1) of Proposition 4.2 and from (e2) for $(\alpha\text{-pr}\,\mathcal{N}(e))_{\alpha \in L_0}$, we get that (u2) of Proposition 4.2 is fulfilled.

From (e3) and (e4) for $(\alpha\text{-pr}\,\mathcal{N}(e))_{\alpha \in L_0}$, we have for all $\alpha \in L_0$ and all $u \in \mathcal{U}^l_\alpha$ that

$$u(x, x) = (f \wedge f^{-1})(x^{-1}x) = (f \wedge f^{-1})(e) \geq \alpha$$

for some $f \in \alpha\text{-pr}\,\mathcal{N}(e)$. Hence, (u3) of Proposition 4.2 holds.

For all $\alpha \in L_0$ and all $u \in \mathcal{U}^l_\alpha$, we have for all $x, y \in G$ that

$$u^{-1}(x, y) = u(y, x) = (f \wedge f^{-1})(y^{-1}x)$$

for some $f \in \alpha\text{-pr}\,\mathcal{N}(e)$. Since (3.4) implies, for all $x, y \in G$, that

$$(f \wedge f^{-1})(x^{-1}y) = f(x^{-1}y) \wedge f^{-1}(x^{-1}y) = f^{-1}(y^{-1}x) \wedge f(y^{-1}x) = (f \wedge f^{-1})(y^{-1}x),$$

that is, $u(x, y) = u(y, x)$ for all $x, y \in G$, then $u \in \mathcal{U}^l_\alpha$ if and only if $u^{-1} \in \mathcal{U}^l_\alpha$ and thus (u4) of Proposition 4.2 holds.

From (e5) for $(\alpha\text{-pr}\,\mathcal{N}(e))_{\alpha \in L_0}$, we have for all $\alpha \in L_0$ and all $f \in \alpha\text{-pr}\,\mathcal{N}(e)$ that there exists $g \in \beta\text{-pr}\,\mathcal{N}(e)$, $\beta \in L_0$, such that $gg \leq f$. For any $u \in \mathcal{U}^l_\alpha$ and all $x, y \in G$, we have $u(x, y) = (f \wedge f^{-1})(x^{-1}y)$ for some $f \in \alpha\text{-pr}\,\mathcal{N}(e)$, which means that there exists $v \in \mathcal{U}^l_\beta$, $\beta \in L_0$, such that (4.2) implies for all $x, y \in G$ that:

$$\begin{aligned}
(v \circ v)(x, y) &= \bigvee_{z \in G} (v(x, z) \wedge v(z, y)) \\
&= \bigvee_{z \in G} ((g \wedge g^{-1})(x^{-1}z) \wedge (g \wedge g^{-1})(z^{-1}y)) \\
&\leq (f \wedge f^{-1})(x^{-1}y) \\
&= u(x, y).
\end{aligned}$$

Hence, by means of (e5) for $(\alpha\text{-pr}\,\mathcal{N}(e))_{\alpha\in L_0}$, we get

$$\alpha \leq \bigvee_{v\in\mathcal{U}^l_\beta,\,(v\circ v)\leq u} \beta \;=\; \bigvee_{g\in\beta\text{-pr}\,\mathcal{N}(e),\,gg\leq f} \beta$$

and then (u5) of Proposition 4.2 holds.

Now, we have the family $(\mathcal{U}^l_\alpha)_{\alpha\in L_0}$ is a family of prefilters on $G\times G$ and fulfills the conditions (u1) - (u5). From Proposition 4.2, we get that $(\mathcal{U}^l_\alpha)_{\alpha\in L_0}$ corresponds a L- uniform structure $\mathcal{U}^l$ on G. This correspondence is given by

$$\mathcal{U}^l(u) = \bigvee_{v\in\mathcal{U}^l_\alpha,\,v\leq u} \alpha \;\text{ and }\; \mathcal{U}^l_\alpha = \alpha\text{-pr}\,\mathcal{U}^l.$$

The same proof can be done with the family $(\mathcal{U}^r_\alpha)_{\alpha\in L_0}$. □

Definition 4.1 $\mathcal{U}^l$ and $\mathcal{U}^r$ defined by (4.5) and (4.6) are called *left* L- uniform structure and *right* L- uniform structure on G, respectively.

An L-topological group (G,τ) is called *abelian* if the group G is abelian.

Proposition 4.4 *For abelian L-topological groups, the left and the right L- uniform structures coincide.*

Proof. Since

$$(f\wedge f^{-1})(x^{-1}y) = (f\wedge f^{-1})(y^{-1}x) = (f\wedge f^{-1})(xy^{-1})$$

for all $x,y\in G$ and for some $f\in\alpha\text{-pr}\,\mathcal{N}(e)$, then $\mathcal{U}^l_\alpha = \mathcal{U}^r_\alpha$ for all $\alpha\in L_0$. Therefore, $\mathcal{U}^l = \mathcal{U}^r$. □

Let $\mathcal{U}$ be a L- filter on $X\times X$ such that $(x,x)^\bullet \leq \mathcal{U}$ holds for all $x\in X$, and let $\mathcal{M}$ be a L- filter on X. Then the mapping $\mathcal{U}[\mathcal{M}] : L^X \to L$, defined by

$$\mathcal{U}[\mathcal{M}](f) \;=\; \bigvee_{u[g]\leq f} (\mathcal{U}(u)\wedge\mathcal{M}(g)) \tag{4.7}$$

for all $f \in L^X$, is a L- filter on X, called the image of $\mathcal{M}$ with respect to $\mathcal{U}$ ([12]), where $u \in L^{X\times X}$ and $g, u[g] \in L^X$ such that:

$$u[g](x) = \bigvee_{y\in X} (g(y) \wedge u(y,x)). \tag{4.8}$$

Proposition 4.5 [12] *Let $\mathcal{U}$ be a L- filter on $X \times X$ such that $(x,x)^{\bullet} \leq \mathcal{U}$ holds for all $x \in X$, and let $\mathcal{M}$ be a L- filter on X. Then the family $(\mathcal{L}_\alpha)_{\alpha\in L_0}$ with*

$$\mathcal{L}_\alpha \;=\; \{\, f \in L^X \mid u[g] \leq f \;\; \textit{for some} \;\; u \in \alpha\text{-pr}\,\mathcal{U} \;\; \textit{and} \;\; g \in \alpha\text{-pr}\,\mathcal{M}\,\}$$

is a valued L- filter base of $\mathcal{U}[\mathcal{M}]$, which consists of prefilters on X such that $\alpha \leq \beta$ implies $\mathcal{L}_\alpha \supseteq \mathcal{L}_\beta$ for all $\alpha, \beta \in L_0$.

Remark 4.1 From Proposition 4.5, we get for a L- uniform structure $\mathcal{U}$ on X and a homogeneous L- filter $\dot{x}$ at $x \in X$, that the family $(\mathcal{L}_\alpha)_{\alpha\in L_0}$ with

$$\mathcal{L}_\alpha \;=\; \{\, f \in L^X \mid u[g] \leq f \;\; \text{for some} \;\; u \in \alpha\text{-pr}\,\mathcal{U} \;\; \text{and} \;\; \alpha \leq g(x)\} \tag{4.9}$$

is a valued L- filter base of $\mathcal{U}[\dot{x}]$, and moreover $\mathcal{L}_\alpha = \alpha\text{-pr}\,\mathcal{U}[\dot{x}]$ for all $\alpha \in L_0$.

To each L- uniform structure $\mathcal{U}$ on X is associated a stratified L- topology $\tau_{\mathcal{U}}$. The related interior operator $\text{int}_{\mathcal{U}}$ is given by ([12]):

$$(\text{int}_{\mathcal{U}} f)(x) \;=\; \mathcal{U}[\dot{x}](f) \tag{4.10}$$

for all $x \in X, f \in L^X$. A L- set $f \in L^X$ is called a *$\tau_{\mathcal{U}}$-neighborhood* of $x \in X$ provided $\alpha \leq \text{int}_{\mathcal{U}} f(x)$ for some $\alpha \in L_0$.

In the following proposition, we show that every L-topological group is uniformizable.

Proposition 4.6 *Any L-topological group (G,τ) is uniformizable. That is, $\tau_{\mathcal{U}^l} = \tau_{\mathcal{U}^r} = \tau_{(\mathcal{U}^l \vee \mathcal{U}^r)} = \tau$.*

Proof. From Lemma 4.1 and Proposition 4.3, we get that both of $\mathcal{U}^l, \mathcal{U}^r$ and $\mathcal{U}^l \vee \mathcal{U}^r$ are L- uniform structures on G.

Since for all $x \in G$ and all $f \in L^G$ we have, from (4.7), (4.10) and Remark 4.1, that:

$$\mathrm{int}_{\mathcal{U}^l} f(x) = \mathcal{U}^l[\dot{x}](f) = \bigvee_{u[g] \leq f} (\mathcal{U}^l(u) \wedge g(x)) = 1$$

is equivalent to

$$\mathrm{int}_{\mathcal{U}^r} f(x) = \mathcal{U}^r[\dot{x}](f) = \bigvee_{u[g] \leq f} (\mathcal{U}^r(u) \wedge g(x)) = 1$$

equivalent to

$$\mathrm{int}_{(\mathcal{U}^l \vee \mathcal{U}^r)} f(x) = (\mathcal{U}^l \vee \mathcal{U}^r)[\dot{x}](f) = \bigvee_{u[g] \leq f} ((\mathcal{U}^l \vee \mathcal{U}^r)(u) \wedge g(x)) = 1,$$

which means that f is a $\tau_{\mathcal{U}^l}$-neighborhood of an element x if and only if it is a $\tau_{\mathcal{U}^r}$-neighborhood of x if and only if it is a $\tau_{(\mathcal{U}^l \vee \mathcal{U}^r)}$-neighborhood of x. Hence, $\tau_{\mathcal{U}^l} = \tau_{\mathcal{U}^r} = \tau_{(\mathcal{U}^l \vee \mathcal{U}^r)}$.

From (4.7) and (4.8), and also from Remark 4.1, we have

$$\begin{aligned} \mathcal{U}^l[\dot{x}](f) &= \bigvee_{g \in \alpha\text{-pr}\mathcal{U}^l[\dot{x}],\, g \leq f} \alpha \\ &= \bigvee_{u[g] \leq f} (\mathcal{U}^l(u) \wedge g(x)) \\ &= \bigvee_{h \in \alpha\text{-pr}\mathcal{N}(x),\, h \leq f} \alpha \\ &= \mathcal{N}(x)(f) \end{aligned}$$

for all $x \in G$ and all $f \in L^G$. Hence, the L- neighborhood filter $\mathcal{U}^l[\dot{x}]$ of $(G, \tau_{\mathcal{U}^l})$ at every $x \in G$ is identical with the L- neighborhood filter $\mathcal{N}(x)$ at every x in the L-topological group (G, τ). Thus, $\tau_{\mathcal{U}^l} = \tau$, and therefore (G, τ) is uniformizable. □

In the following we show that these conditions (e1) - (e5) for a family of prefilters on G are also sufficient to construct from the group G an L-topological group.

Proposition 4.7 *Let G be a group and e the identity element of G, and let $(\mathcal{V}^e_\alpha)_{\alpha\in L_0}$ be a family of prefilters on G fulfilling conditions* (e1) - (e5)*. Defining, for each $\alpha \in L_0$, the subsets*

$$\mathcal{U}^l_\alpha = \{u \in L^{G\times G} \mid u(x,y) = (f \wedge f^{-1})(x^{-1}y) \text{ for some } f \in \mathcal{V}^e_\alpha\}$$

and

$$\mathcal{U}^r_\alpha = \{u \in L^{G\times G} \mid u(x,y) = (f \wedge f^{-1})(xy^{-1}) \text{ for some } f \in \mathcal{V}^e_\alpha\}$$

of $L^{G\times G}$. Hence, we have the left and the right L- uniform structures $\mathcal{U}^l$ and $\mathcal{U}^r$ on G defined by (4.5) and (4.6)*, respectively. Moreover, $\tau_{\mathcal{U}^l} = \tau_{\mathcal{U}^r} = \tau_{(\mathcal{U}^l\vee\mathcal{U}^r)}$ is a L- topology τ on G for which the pair (G,τ) is an L-topological group. Finally, for each $\alpha \in L_0$, we have $\mathcal{V}^e_\alpha = \alpha\text{-pr}\mathcal{N}(e)$, where $\mathcal{N}(e)$ is the L- neighborhood filter at e with respect to the L- topology τ on G.*

Proof. As in Propositions 4.3 and 4.6, we get that $\mathcal{U}^l$ and $\mathcal{U}^r$ are the left and the right L- uniform structures on G for which $\tau_{\mathcal{U}^l} = \tau_{\mathcal{U}^r} = \tau_{(\mathcal{U}^l\vee\mathcal{U}^r)}$ is a L- topology on the group G. Denote $\tau_{\mathcal{U}^l} = \tau_{\mathcal{U}^r} = \tau_{(\mathcal{U}^l\vee\mathcal{U}^r)}$ by τ.

It remains to prove that (G,τ) is an L-topological group and that $\mathcal{V}^e_\alpha = \alpha\text{-pr}\mathcal{N}(e)$ for all $\alpha \in L_0$.

Now, from that the conditions of Proposition 2.1 are equivalent to the conditions (e1) - (e3), we get that

$$\mathcal{V}^e_\alpha = \alpha\text{-pr}\,\mathcal{U}^l[\dot{e}] = \alpha\text{-pr}\,\mathcal{U}^r[\dot{e}] = \alpha\text{-pr}\,(\mathcal{U}^l \vee \mathcal{U}^r)[\dot{e}]$$

for all $\alpha \in L_0$. That is, $\mathcal{V}^e_\alpha = \alpha\text{-pr}\,\mathcal{N}(e)$ for all $\alpha \in L_0$, where $\mathcal{N}(e)$ is the L-neighborhood filter of (G,τ) at e.

From conditions (e4) and (e5) of the prefilters $\alpha\text{-pr}\,\mathcal{N}(e)$ for all $\alpha \in L_0$, we get that for all $f \in \alpha\text{-pr}\,\mathcal{N}(e)$, there exist $g, h \in \alpha\text{-pr}\,\mathcal{N}(e)$ for some $\alpha \in L_0$ such that $g^{-1}h \leq f$, which means that

$$(ga)^{-1}(hb) = a^{-1}(g^{-1}h)b \leq a^{-1}fb.$$

That is, from Lemma 3.2, we get that for all $\lambda = a^{-1}fb \in \alpha\text{-pr}\,\mathcal{N}(a^{-1}b)$, there exist $\mu = ga \in \alpha\text{-pr}\,\mathcal{N}(a)$ and $\nu = hb \in \alpha\text{-pr}\,\mathcal{N}(b)$ such that $\mu^{-1}\nu \leq \lambda$. Hence, (G, τ) is an L-topological group. □

Let us define the following.

Definition 4.2 Let $\mathcal{U}$ be a L- uniform structure on a set X. Then

(1) $u \in L^{X \times X}$ is called a *surrounding* provided $\mathcal{U}(u) \geq \alpha$ for some $\alpha \in L_0$ and $u = u^{-1}$,

(2) A surrounding $u \in L^{X \times X}$ is called *left (right) invariant* provided

$$u(ax, ay) = u(x, y) \qquad (u(xa, ya) = u(x, y)) \text{ for all } a, x, y \in X,$$

(3) $\mathcal{U}$ is called a *left (right) invariant* L- uniform structure if $\mathcal{U}$ has a valued L-filter base consists of left (right) invariant surroundings.

Now, from Proposition 4.3, we have this remark.

Remark 4.2 In the L-topological group (G, τ), for each element u in $\mathcal{U}^l_\alpha$, defined by (4.3), we have $\mathcal{U}^l_\alpha(u) \geq \alpha$ for some $\alpha \in L_0$ and also, for all $x, y \in G$ and each $u \in \mathcal{U}^l_\alpha$, we have

$$\begin{aligned} u(x, y) &= (f \wedge f^{-1})(x^{-1}y) \text{ for some } f \in \alpha\text{-pr}\,\mathcal{N}(e) \\ &= (f \wedge f^{-1})(y^{-1}x) \text{ for some } f \in \alpha\text{-pr}\,\mathcal{N}(e) \\ &= u(y, x) \\ &= u^{-1}(x, y). \end{aligned}$$

That is, $\mathcal{U}^l_\alpha$ is a prefilter of surroundings. Also, for all $a, x, y \in G$, we have

$$\begin{aligned} u(ax, ay) &= (f \wedge f^{-1})((ax)^{-1}(ay)) \text{ for some } f \in \alpha\text{-pr}\mathcal{N}(e) \\ &= (f \wedge f^{-1})(x^{-1}y) \text{ for some } f \in \alpha\text{-pr}\mathcal{N}(e) \\ &= u(x, y) \text{ for all } u \in \mathcal{U}^l_\alpha \text{ and for all } x, y \in G. \end{aligned}$$

Thus, the elements of $\mathcal{U}^l_\alpha$ are left invariant surroundings. Moreover, $(\mathcal{U}^l_\alpha)_{\alpha\in L_0}$ is a valued L- filter base for the left L- uniform structure $\mathcal{U}^l$ defined by (4.5), and hence $\mathcal{U}^l$ is a left invariant L- uniform structure on G. By the same way, $\mathcal{U}^r$, defined by (4.6), is a right invariant L- uniform structure on G.

Notice that: Between any two systems of sets $\mathcal{A}$ and $\mathcal{B}$, we recall that $\mathcal{A}$ is called *coarser than* $\mathcal{B}$ if for any $A \in \mathcal{A}$, there is $B \in \mathcal{B}$ such that $B \subseteq A$.

The following important proposition is now obtained from our last results.

Proposition 4.8 *Let (G, τ) be an L-topological group. Then there exist on G a unique left invariant L- uniform structure $\mathcal{U}^l$ and a unique right invariant L- uniform structure $\mathcal{U}^r$ compatible with τ, constructed in* Proposition 4.3 *using the family $(\alpha\text{-pr}\,\mathcal{N}(e))_{\alpha\in L_0}$ of all prefilters α-pr$\,\mathcal{N}(e)$, where $\mathcal{N}(e)$ is the L- neighborhood filter at the identity element e of the L-topological group (G, τ).*

Proof. From Propositions 4.3 and 4.6, and Remark 4.2, we have $\mathcal{U}^l$ and $\mathcal{U}^r$ are the left and the right invariant L- uniform structures on G, respectively for which $\tau_{\mathcal{U}^l} = \tau_{\mathcal{U}^r} = \tau$.

Suppose that $(\mathcal{V}^l_\alpha)_{\alpha\in L_0}$ is a valued L- filter base for a left invariant L- uniform structure $\mathcal{V}^l$ on G such that $\tau_{\mathcal{V}^l} = \tau_{\mathcal{U}^l} = \tau$.

Since for any $v_1 \in \mathcal{V}^1_\alpha$, there exists $v_2 \in \mathcal{V}^1_\alpha$ with $v_2 \leq v_1$ and $v_2(ax, ay) = v_2(x, y)$ for all $a, x, y \in G$. From (4.8), we get that $v_2[e_1](x) = v_2(e, x)$ for all $x \in G$, that is, $v_2[e_1](e) = v_2(e, e) \geq \alpha$ and there exists a left invariant surrounding $u \in \mathcal{U}^l_\alpha$ such that $u[e_1] \leq v_2[e_1]$.

Now, $u(x, y) = u(xx^{-1}, x^{-1}y) = u(e, x^{-1}y) = u[e_1](x^{-1}y) \leq v_2[e_1](x^{-1}y)$ for all $x, y \in G$, which means that $u(x, y) = v_2(e, x^{-1}y) = v_2(x, y)$ and also we have $v_2 \leq v_1$, so $u \leq v_1$. That is, for all $\alpha \in L_0$ and for any $v_1 \in \mathcal{V}^l_\alpha$, there exists $u \in \mathcal{U}^l_\alpha$ such that $u \leq v_1$, and this means that $\mathcal{V}^l_\alpha$ is coarser than $\mathcal{U}^l_\alpha$ for all $\alpha \in L_0$. By the same way, we can show that $\mathcal{U}^l_\alpha$ is coarser than $\mathcal{V}^l_\alpha$ for all $\alpha \in L_0$, and thus $\mathcal{V}^l_\alpha = \mathcal{U}^l_\alpha$ for all $\alpha \in L_0$. Hence, $\mathcal{V}^l = \mathcal{U}^l$.

Similarly, one can prove that the right invariant L- uniform structure $\mathcal{U}^r$ is unique. □

5. The relation between the L-topological groups and the $GT_{3\frac{1}{2}}$-spaces

In this section we shall show and prove the relation between our notion of $GT_{3\frac{1}{2}}$-spaces and the notion of L-topological groups defined in [1].

In [2, 3, 5] we had defined the L- separation axioms GT_i, $i = 0, 1, 2, 3, 3\frac{1}{2}, 4$. Here, we recall some of these axioms which we need in the following.

A L- topological space (X, τ) is called ([2, 3, 5]):

(1) GT_0 if for all $x, y \in X$ with $x \neq y$ we have $\dot{x} \not\leq \mathcal{N}(y)$ or $\dot{y} \not\leq \mathcal{N}(x)$.

(2) GT_1 if for all $x, y \in X$ with $x \neq y$ we have $\dot{x} \not\leq \mathcal{N}(y)$ and $\dot{y} \not\leq \mathcal{N}(x)$.

(3) GT_2 if for all $x, y \in X$ with $x \neq y$ we have $\mathcal{N}(x) \wedge \mathcal{N}(y)$ does not exist.

(4) GT_3 if it is GT_1 and if for all $x \in X$ and all $F \in \tau'$ with $x \notin F$, we have $\mathcal{N}(x) \wedge \mathcal{N}(F)$ does not exist.

(5) *completely regular* if for all $x \notin F \in \tau'$, there exists a L- continuous mapping $f : (X, \tau) \to (I_L, \Im)$ such that $f(x) = \overline{1}$ and $f(y) = \overline{0}$ for all $y \in F$.

(6) $GT_{3\frac{1}{2}}$ (or L-*Tychonoff*) if it is GT_1 and completely regular.

(7) GT_4 if it is GT_1 and if for all $F, G \in \tau'$ with $F \cap G = \emptyset$, we have $\mathcal{N}(F) \wedge \mathcal{N}(G)$ does not exist.

Denote by GT_i-*space* the L- topological space which is GT_i, $i = 0, 1, 2, 3, 3\frac{1}{2}, 4$.

Proposition 5.1 [2, 3, 5] *Every GT_i-space is GT_{i-1}-space for each $i = 1, 2, 3, 4$, and $GT_{3\frac{1}{2}}$-spaces fulfill the following:*

$$\textit{every } GT_4\textit{-space} \text{ is a } GT_{3\frac{1}{2}}\textit{-space} \text{ and } \textit{every } GT_{3\frac{1}{2}}\textit{-space} \text{ is a } GT_3\textit{-space.}$$

Proposition 5.2 [6] *If $\mathcal{U}$ is a L- uniform structure on a set X and $\tau_{\mathcal{U}}$ the L-topology associated to $\mathcal{U}$, then $(X, \tau_{\mathcal{U}})$ is a completely regular space.*

The fact that the L- topology of an L-topological group can be induced by a left or right invariant L- uniform structure leads us to our fundamental results in this section as follows.

Proposition 5.3 *The L- topology of an L-topological group is completely regular.*

Proof. The proof goes directly from Propositions 4.6 and 5.2 □

Definition 5.1 An L-topological group (G, τ) is called *separated* if for the identity element e, we have $\bigwedge_{f \in \alpha\text{-pr}\mathcal{N}(e)} f(e) \geq \alpha$, and $\bigwedge_{f \in \alpha\text{-pr}\mathcal{N}(e)} f(x) < \alpha$ for all $x \in G$ with $x \neq e$ and for all $\alpha \in L_0$.

A L- uniform structure $\mathcal{U}$ on a set X is called *separated* ([4]) if for all $x, y \in X$ with $x \neq y$ there is $u \in L^{X \times X}$ such that $\mathcal{U}(u) = 1$ and $u(x, y) = 0$. The space $(X, \mathcal{U})$ is called *separated L- uniform space.*

Proposition 5.4 [4] *Let X be a set, $\mathcal{U}$ a L- uniform structure on X and $\tau_{\mathcal{U}}$ the L-topology associated with $\mathcal{U}$. Then*

$$(X, \mathcal{U}) \textit{ is separated} \text{ if and only if } (X, \tau_{\mathcal{U}}) \textit{ is } GT_0\textit{-space.}$$

In the following result we have shown the expected relation between our notion of $GT_{3\frac{1}{2}}$-spaces and the notion of L-topological groups.

Proposition 5.5 *Let (G,τ) be an L-topological group. Then the following statements are equivalent.*

(1) *The L- topology τ is GT_0.*

(2) *The L- topology τ is GT_1.*

(3) *The L- topology τ is GT_2.*

(4) *The L- topology τ is $GT_{3\frac{1}{2}}$.*

(5) *$\mathcal{U}^l$ is separated.*

(6) *$\mathcal{U}^r$ is separated.*

(7) *The L-topological group (G,τ) is separated.*

Proof.

(1) $\Rightarrow$ (2): Let $x \neq y$ in G, then for one point (say x) there exists a τ-neighborhood f such that $\mathrm{int}_\tau f(x) \geq \alpha > f(y)$, which means that there is $h \in \alpha$-pr$\mathcal{N}(e)$ such that $h = x^{-1}f$ and then $k = h \wedge h^{-1}$ is a symmetric τ-neighborhood of e, and this means that the L- set $g = yk$ is a τ-neighborhood of y for which $\mathrm{int}_\tau g(y) \geq \alpha > g(x)$ because if otherwise $g(x) = yk(x) \geq \alpha$, then

$$\alpha \leq g^{-1}(x^{-1}) = (h \wedge h^{-1})y^{-1}\,(x^{-1}) = (x^{-1}f \wedge f^{-1}x)y^{-1}\,(x^{-1}) \leq x^{-1}fy^{-1}\,(x^{-1}),$$

that is, $fy^{-1}(e) \geq \alpha$, and then $f(y) \geq \alpha$ which is a contradiction. Hence there exists a τ-neighborhood g of y such that $\mathrm{int}_\tau g(y) \geq \alpha > g(x)$, and thus (G,τ) is a GT_1-space.

(2) $\Rightarrow$ (3): It is clear from Propositions 5.1 and 5.3.

(3) $\Rightarrow$ (4): Obvious.

(4) $\Rightarrow$ (5) and (4) $\Rightarrow$ (6): The proof comes from Proposition 4.6, and from Propositions 5.1 and 5.4.

(5) $\Rightarrow$ (7): Since $\mathcal{U}^l$ is separated then, by means of Propositions 4.6 and 5.4, $\tau = \tau_{\mathcal{U}^l}$ is GT_0. Thus for any $x \neq e$ in G, there exists $f \in \alpha\text{-pr}\mathcal{N}(e)$ such that $f(x) < \alpha \leq \text{int}_\tau f(e) \leq f(e)$. Hence, $\bigwedge\limits_{f\in\alpha\text{-pr}\mathcal{N}(e)} f(x) \geq \alpha$ whenever $x = e$ and $\bigwedge\limits_{f\in\alpha\text{-pr}\mathcal{N}(e)} f(x) < \alpha$ otherwise. That is, (G, τ) is a separated L-topological group.

(6) $\Rightarrow$ (7): The proof goes similar to the case (5) $\Rightarrow$ (7).

(7) $\Rightarrow$ (1): If $x, y \in G$ with $x \neq y$, then $x^{-1}y \neq e$ and then $\bigwedge\limits_{f\in\alpha\text{-pr}\mathcal{N}(e)} f(x^{-1}y) < \alpha$, which means that there exists $f \in \alpha\text{-pr}\mathcal{N}(e)$ such that $f(x^{-1}y) < \alpha$, that is, $xf\,(y) = \bigwedge\limits_{f(z)>0} (xz)_1\,(y) < \alpha$, where $z = x^{-1}y$ is not allowed. Since $\{xf \mid f \in \alpha\text{-pr}\mathcal{N}(e)\}$ is itself $\alpha\text{-pr}\mathcal{N}(x)$, that is, the set of all α-L- neighborhoods of x and $xf\,(y) < \alpha$. Hence, $xf\,(y) < \alpha \leq \text{int}_\tau(xf)\,(x)$. Thus, (G, τ) is GT_0. $\square$

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